Available at
www.ComputerScienceWeb.com
POWERED BY SCIENCE DIRECT®

Signal Processing 83 (2003) 2319–2333

SIGNAL PROCESSING

www.elsevier.com/locate/sigpro

A method for modelling and simulation of fractional systems

T. Poinot*, J.-C. Trigeassou

Laboratoire d'Automatique et d'Informatique Industrielle, Ecole Supérieure d'Ingénieurs de Poitiers, 40 Avenue du Recteur Pineau 86 022 Poitiers Cedex, France

Received 19 December 2002

Abstract

An original method for modelling and simulation of fractional systems is presented in this article. The basic idea is to model the fractional system by a state-space representation, where conventional integration is replaced by fractional one with the help of non-integer integrator. This operator is itself approximated by a N dimensional system composed of an integrator and of a phase-lead filter. This method is compared to other techniques like direct discretization of the fractional derivator and diffusive representation. Numerical simulations exhibit the general applicability and flexibility of this new approach to different types of fractional models and to non-conventional non-integer derivation with limited spectral range.

Keywords: Fractional systems; Fractional operator; Modelling; Simulation; State-space representation

1. Introduction

Non-integer order systems, also known as fractional filters, have been introduced long ago in various fields of science such as electrochemistry [7], thermal engineering [1], acoustics [15], electromagnetism [22], etc. where they are fundamentally used for the modelling of diffusion processes. These systems are characterized by long memory transients and infinite dimensional structure. Their dynamics depend on the well-known Diffusion Equation and on the geometry of the considered problem. If the attention is focussed on the relation between variables at the boundary region, a theoretical modelling leads to an integrator with order equal to 0.5. Generalization of this modelling to more complex situations needs the help of a fractional model, characterized by its non-integer order, whose value can vary from 0 and 1. Consequently, the modelling of fractional systems turns out to be necessary for simulation, identification and control [4,8–14,18–20].

This modelling is fundamentally based on non-integer derivation. Then, the numerical simulation of these systems is highly linked to the modelling of the non-integer derivator, or equivalently of the non-integer integrator. Two main approaches are commonly used. A direct solution [19] is based on the discretization of the derivator: then, the fractional model is replaced by a difference equation model, with long memory behaviour. An indirect solution uses Diffusive Representation [6,17], with finite discretization of the continuous fractional model into a N-dimensional state-space representation with conventional integer derivation.

* Corresponding author. Tel.: +33-5-49-45-35-12; fax: +33-5-49-45-40-34.
E-mail addresses: thierry.poinot@esip.univ-poitiers.fr (T. Poinot), jean-claude.trigeassou@esip.univ-poitiers.fr (J.-C. Trigeassou).

0165-1684/$ - see front matter © 2003 Elsevier B.V. All rights reserved.
doi:10.1016/S0165-1684(03)00185-3

In this article, we propose a third approach based on a fractional integration operator in order to simulate a fractional system with a conventional state-space representation approach. Fundamentally, simulation of a state-space model needs integration operators; thus simulation of a fractional model needs also a non-integer integration operator. This new operator is defined in the frequency domain, with reference to the ideal integrator of order n $(0 < n < 1)$. An approximation is necessary, because it is impossible to use an infinite spectral range. Thus the corresponding fractional operator acts with order n on a limited frequency domain and with order one outside this domain. Moreover, this approximation is performed by a finite dimensional system or equivalently by its state-space representation. Then, this fractional integration operator is used to simulate the corresponding system with an appropriate macro state-space representation. Analysis of this operator permits a comparison with the other simulation techniques. We demonstrate that it can be interpreted as a diffusive representation system with special interesting features, permitting a general approach of the simulation of fractional systems, without static errors.

This article is divided in four parts. The first one presents the main methods used in order to simulate fractional systems. The second part is devoted to the definition and the modelling of the fractional integrator. This new operator is used to define the state-space representation of a non-integer system in the third part. Modelling of different types of fractional systems are illustrated by numerical simulations in the fourth part.

2. Simulation of fractional systems

Many methods are used in order to simulate non-integer or fractional systems. Two types of methods can be considered. The first ones, also called direct methods, are based on an numerical approximation of the non-integer derivator operator. The second ones, called indirect methods, are based on the simulation of the continuous fractional model, with the help of a specific operator or representation.

2.1. Direct methods

In these methods the fractional derivator operator is replaced by a numerical approximation, in order to obtain a recurrent equation directly used for simulation. Different types of approximations can be used. The more usual approximation is the one directly related to the definition given by Grünwald [16]:

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(Kh) = \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} f((K-k)h), \tag{1}$$

where h is the sampling period and

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}. \tag{2}$$

In order to illustrate this approach, consider for instance the fractional system defined by

$$\frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + a_0 y(t) = b_0 u(t) \quad 0 < n < 1. \tag{3}$$

This system will be used as a benchmark model for the different techniques of simulation.

Using previous numerical approximation, the following equation is obtained [19]:

$$\sum_{k=0}^{K} \frac{(-1)^k}{h^n} \binom{n}{k} y((K-k)h) + a_0 y(Kh) = b_0 u(Kh), \tag{4}$$

where K is the number of data such as $t = Kh$.

The system output is given by

$$y(Kh) = \frac{b_0 u(Kh) - \sum_{k=1}^{K} \frac{(-1)^k}{h^n} \binom{n}{k} y((K-k)h)}{a_0 + 1/h^n}. \tag{5}$$

This method is very simple to use. On the other hand, the simulation requires, for each step, the computation of sums of increasing dimension with time. Other approximations can be used, refer for instance to [3,23].

2.2. Indirect method based on diffusive representation

Indirect methods refer to the simulation of the continuous fractional model with the help of a specific operator or representation. Diffusive Representation (DR) approach [6,17] is typically one of these methods.

Consider a fractional system, whose impulse response is $h(t)$: it is composed of a weighted sum of an infinity of modes $e^{-\xi_k t}$, where ξ_k varies from very small to very large values.

Thus

$$h(t) = \sum_{k=1}^{K} \mu(\xi_k) e^{-\xi_k t} \Delta\xi_k. \tag{6}$$

Considering $\Delta\xi_k \to 0$ and $K \to \infty$, we get a continuous limit:

$$h(t) = \int_0^{\infty} \mu(\xi) e^{-\xi t}\, d\xi, \tag{7}$$

where $\mu(\xi)$ is the weight density function of $e^{-\xi t}$ modes, it is called Diffusive Representation (DR).

Note that in Eq. (7) $h(t)$ can be interpreted as the transform (Laplace transform L) of $\mu(\xi)$, where $t \equiv s$; $\xi \equiv t$.

Thus, $\mu(\xi)$ can be obtained from $h(t)$ using inverse transform (L^{-1}):

$$h(t) = L\{\mu(\xi)\}, \tag{8}$$

$$\mu(\xi) = L^{-1}\{h(t)\}. \tag{9}$$

The main difference with Laplace transform is that the two variables ξ and t are only real ones.

An infinite dimension state vector representation is attached to each fractional system.

Let $u(t)$ be the input and $y(t)$ the output; then the input/output representation of the system is given by

$$y(t) = h(t) * u(t) \tag{10}$$

or equivalently by

$$Y(s) = H(s)U(s) \tag{11}$$

with

$$H(s) = L\{h(t)\}. \tag{12}$$

Consider Eq. (6) and define

$$c_k = \mu(\xi_k)\Delta\xi_k \tag{13}$$

then

$$H(s) = L\left\{\sum_{k=1}^{K} c_k e^{-\xi_k t}\right\} = \sum_{k=1}^{K} \frac{c_k}{s + \xi_k}. \tag{14}$$

Input/output representation (10) is equivalent to the state vector model derived from Eq. (14):

$$\frac{dx_k(t)}{dt} = -\xi_k x_k(t) + u(t) \quad (k = 1 \text{ to } K), \tag{15}$$

$$y(t) = \sum_{k=1}^{K} c_k x_k(t) = \sum_{k=1}^{K} \mu(\xi_k) x_k(t) \Delta\xi_k. \tag{16}$$

Considering $\Delta\xi_k \to 0$ and $K \to \infty$, the continuous limit is

$$\frac{\partial x(\xi, t)}{\partial t} = -\xi x(\xi, t) + u(t), \tag{17}$$

$$y(t) = \int_0^{\infty} \mu(\xi) x(\xi, t)\, d\xi. \tag{18}$$

Numerical simulation of a fractional system is based on discretization of the variable ξ in Eqs. (17) and (18), that is to say simulation is performed using Eqs. (15) and (16).

Consider previous benchmark model (3):

$$H(s) = \frac{b_0}{a_0 + s^n} \quad 0 < n < 1. \tag{19}$$

It is necessary to use the DR $\mu(\xi)$ corresponding to this model. G. Montseny has demonstrated [17] that:

$$\mu(\xi) = \frac{b_0 \frac{\sin(n\pi)}{\pi} \xi^n}{\xi^{2n} + 2a_0 \frac{\cos(n\pi)}{\pi} \xi^n + a_0^2}. \tag{20}$$

The next step is the choice of the discretization for ξ_k. A geometric progression for ξ_k and $\Delta\xi_k$ is certainly a better choice than the arithmetic one because it reduces the number of computations. Thus we obtain the system (15) and (16) approximating the fractional model.

Finally, numerical integration of this system is the solution of the simulation problem.

Remark: the same approach can be used to simulate a non-integer integrator [6], whose impulse response is

$$h_i(t) = \frac{1}{\Gamma(n)} t^{n-1} \quad t > 0, \ 0 < n < 1 \tag{21}$$

and transfer function is

$$I(s) = \frac{1}{s^n}. \tag{22}$$

This fractional integrator is characterized by the following DR [17]:

$$\mu(\xi) = \frac{\sin(n\pi)}{\pi}\,\xi^{-n}. \tag{23}$$

By discretization, a state space vector representation is attached to $I(s)$, identical by its form to (15) and (16), differing essentially by the choice of c_k weight. Fundamentally, the fractional integrator is approximated by:

$$I(s) = \sum_{k=1}^{K} \frac{c_k}{s + \xi_k}, \tag{24}$$

where the weights c_k are related to the DR (23).

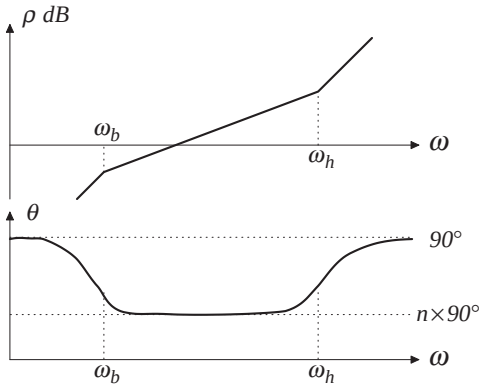

Fig. 1. Bode diagram of the fractional derivator.

3. A fractional Integrator operator

3.1. Fractional derivator

There are many ways to define the ideal fractional derivator [19]. Using the transform technique, let $D(s) = s^n$ be the Laplace transform of $(\mathrm{d}^n/\mathrm{d}t^n)$, where $0 < n < 1$. Then $D(\mathrm{j}\omega)$ is the Fourier transform defined by

$$D(\mathrm{j}\omega) = \rho \mathrm{e}^{\mathrm{j}\theta} \qquad \rho = \omega^n, \;\; \theta = n\,\frac{\pi}{2}. \tag{25}$$

With this definition, the fractional domain where $D(\mathrm{j}\omega)$ acts as a non-integer derivator ranges from 0 to ∞. A. Oustaloup [18,19] has shown that the spectral range has to be necessarily limited to $[\omega_b, \omega_h]$.

Moreover, experiments show that this spectral range can be very limited, for instance over one decade when behaviour is caused by an artificial diffusion process [19]. On the contrary, when the domain $[\omega_b, \omega_h]$ is very large, there is not a great difference with an ideal fractional derivator. Thus, for narrow band derivators, it is necessary to define in a realistic way the behaviour of the derivator outside the limited domain.

So, let us consider the Bode plot of the proposed practical derivator in Fig. 1.

It is composed of three parts:

- The intermediate part corresponds to non-integer action, characterized by the order n $(0 < n < 1)$.
- In the other two parts, the derivator has conventional action, characterized by its order 1.

In this way, we define a new operator $D_n(\mathrm{j}\omega)$ $(D_n(s))$ which is a conventional derivator, except on a limited band $[\omega_b, \omega_h]$ where it acts like a "n" non-integer derivator. Thus, this operator $D_n(s)$ is characterized by three parameters: ω_b, ω_h and n.

3.2. Fractional integrator

Practically, simulation of systems is not performed with derivators but with integrators. It is easy to define the practical non-integer integrator as the inverse of $D_n(s)$, thus

$$I_n(s) = \frac{1}{D_n(s)}. \tag{26}$$

So, our interest will be focussed on the modelling and simulation of the operator $I_n(s)$, whose Bode plot is dual of that of $D_{\mathrm{j}\omega}(s)$. This Bode diagram can be obtained using a transfer function like:

$$I_n(s) = \frac{G_n}{s}\left(\frac{1 + (s/\omega_b)}{1 + (s/\omega_h)}\right)^n. \tag{27}$$

The synthesis of this operator is performed by the association of an integrator $1/s$ and of the conventional phase lead filter used by Oustaloup [19]:

$$A_v(\mathrm{j}\omega) = \prod_{i=1}^{N} \frac{1 + \mathrm{j}(\omega/\omega_i')}{1 + \mathrm{j}(\omega/\omega_i)} \tag{28}$$

composed of N cells and characterized by four design parameters ω_i', ω_i, α, η (n' is derived from α and η

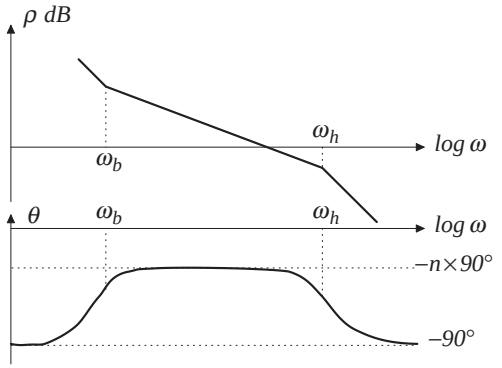

Fig. 2. Bode diagram of the fractional integrator.

according to (29)). Let us define:

- ω'_1: the lower pulsation,
- ω_N: the higher one.

with, for the ith cell:

$$\omega_i = \alpha\omega'_i \quad \text{with } \alpha > 1$$

$$\omega'_{i+1} = \eta\omega_i \quad \text{with } \eta > 1.$$

If N is sufficiently high, the Bode plot of $A_v(j\omega)$, inside $[\omega'_1, \omega_N]$, is characterized by a positive slope equal to $n' \times 20$ dB/dec and a constant positive phase equal to $n' \times 90°$, where

$$n' = \frac{\log \alpha}{\log \alpha\eta}. \tag{29}$$

Thus, combining $A_v(j\omega)$ with conventional integrator $1/j\omega$, we obtain the Bode plot of Fig. 2.

This means that the operator $I_n(s)$ can be approximated by

$$I_n(s) = \frac{G_n}{s} \prod_{i=1}^{N} \frac{1 + s/\omega'_i}{1 + s/\omega_i}. \tag{30}$$

This operator is characterized by six parameters:

- ω'_1 and ω_N define the frequency range (equivalently to ω_b and ω_h),
- N is the number of cells (it is directly related to the quality of the desired approximation),
- α and η are recursive parameters related to non-integer order n and
- G_n is defined in order to have the same gain for $1/s^n$ and $I_n(s)$ at the pulsation $\omega_u = 1$ rd/s.

This operator is completely defined by the following relations:

$$\omega_i = \alpha\omega'_i, \quad \omega'_{i+1} = \eta\omega_i, \quad n = 1 - \frac{\log \alpha}{\log \alpha\eta}. \tag{31}$$

3.3. State-space representation of the operator

It is necessary to associate a state-space representation to $I_n(s)$ in order to simulate more complex systems. An infinity of these representations can be associated to $I_n(s)$. Because one of our objectives is to estimate the parameters $\{\omega'_1, \omega_N, \alpha \text{ and } \eta\}$, we have privileged parsimonious models in order to facilitate the identification procedure [10,12–14,20].

Because $I_n(s)$ is composed of a product of cells, we have defined the state variables as the output of each cell, according to Fig. 3.

With this "natural" decomposition, each state variable x_n is only related to the preceding x_{n-1} by

$$-\frac{\omega_{n-1}}{\omega'_{n-1}}\dot{x}_{n-1} + \dot{x}_n = \omega_{n-1}(x_{n-1} - x_n), \tag{32}$$

where $\frac{\omega_{n-1}}{\omega'_{n-1}} = \alpha$.

Considering x_{n+1}, we obtain

$$-\alpha\dot{x}_n + \dot{x}_{n+1} = \omega_n(x_n - x_{n+1}). \tag{33}$$

The connection between these two rows (n and $n-1$) is realized by

$$\omega_n = \alpha\eta\omega_{n-1}.$$

Thus, this particular state-space representation uses only α and η, plus ω'_1 or ω_N.

So, we can write:

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -\alpha & 1 & & & \vdots \\ 0 & -\alpha & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\alpha & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \vdots \\ \dot{x}_{N+1} \end{bmatrix}$$

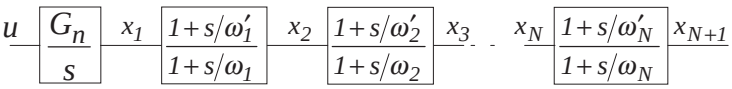

Fig. 3. $I_n(s)$ block diagram.

$$= \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \omega_1 & -\omega_1 & & & \vdots \\ 0 & \omega_2 & -\omega_2 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \omega_N & -\omega_N \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{N+1} \end{bmatrix}$$

$$+ \begin{bmatrix} G_n \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} u.$$

Then, defining M_I, A_I, $\underline{B}_I$ and $\underline{x}_I$, we can write (where M_I, A_I and $\underline{B}_I$ are parsimonious matrixes):

$$M_I \dot{\underline{x}}_I = A_I \underline{x}_I + \underline{B}_I u \tag{34}$$

or equivalently:

$$\dot{\underline{x}}_I = A_I^* \underline{x}_I + \underline{B}_I^* u, \tag{35}$$

where

$$A_I^* = M_I^{-1} A_I,$$

$$\underline{B}_I^* = M_I^{-1} \underline{B}_I$$

are full matrixes necessary for the numerical simulation of the operator.

Refer to Fig. 4 for a symbolic representation of this state-space model.

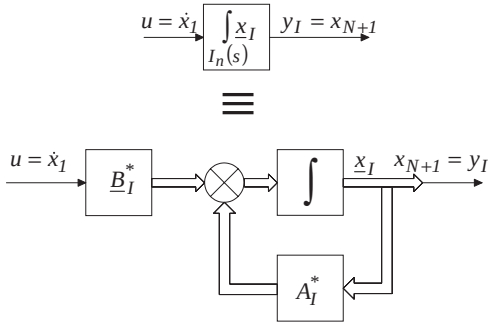

Fig. 4. State-space representation of the operator.

4. State-space representation of a non-integer system

4.1. Principle

Using the fractional integrator, one can construct the state-space representation of general fractional systems. We will again consider the benchmark model (3):

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{a_0 + s^n}. \tag{36}$$

This system has an aperiodic response if $0 < n < 1$ and an oscillatoring one if $1 < n < 2$.

This transmittance is equivalent to the differential equation form:

$$\frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + a_0 y(t) = b_0 u(t). \tag{37}$$

Let us define $x(t)$ such as

$$X(s) = \frac{1}{s^n + a_0} U(s). \tag{38}$$

Thus, we obtain a "macro" state-space representation of this system (with "macro" parameters a_0 and b_0).

$$\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} = -a_0 x(t) + u(t),$$

$$y(t) = b_0 x(t), \tag{39}$$

or equivalently using the operator of part 3:

$$\begin{cases} \dot{x}_1 = G_n(-a_0 x_{N+i} + u) \\ y = b_0 x_{N+i} \end{cases} \quad \text{with}$$

$$\begin{cases} i = 1 & \text{if } 0 < n < 1 \\ i = 2 & \text{if } 1 < n < 2. \end{cases} \tag{40}$$

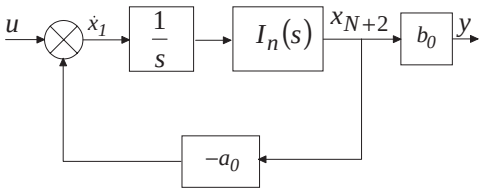

Fig. 5. State-space representation of the system.

Fig. 5 represents the block diagram of the system for $1 < n < 2$ (in the case where $0 < n < 1$, it is necessary to remove the first block integrator $1/s$).

Remark: This system is stable for $0 < n < 2$, unstable for $n \geqslant 2$.

4.2. Analysis of the state-space model

This "macro" model (40) is only convenient for compact writing. Practically, there are two imbricated state-space representations, one for the "macro" model, the other for the integrator operator, as represented in Fig. 5.

Because $\dot{\underline{x}}_I = -a_0 x_{N+1} + u$, we obtain for the global state-space model:

$$M\dot{\underline{x}} = A\underline{x} + \underline{B}u, \quad y = \underline{C}^{\mathrm{T}}\underline{x} \tag{41}$$

with

- if $0 < n < 1$

$$A = A_I + \begin{bmatrix} 0 & \cdots & 0 & -G_n a_0 \\ \vdots & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad \underline{B} = \underline{B}_I$$

$$M = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\alpha & 1 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & -\alpha & 1 \end{bmatrix},$$

$$\underline{C}^{\mathrm{T}} = [0 \quad \cdots \quad 0 \quad b_0],$$

- if $1 < n < 2$

$$A = \begin{bmatrix} 0 & \cdots & 0 & -G_n a_0 \\ \vdots & & & \\ \vdots & & A_I & \\ 0 & & & \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} \underline{B}_I \\ 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & & & \vdots \\ 0 & -\alpha & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\alpha & 1 \end{bmatrix},$$

$$\underline{C}^{\mathrm{T}} = [0 \quad \cdots \quad 0 \quad b_0].$$

Remarks:

- N has to be very large in order to perform an appropriate approximation of the fractional integrator. Then, $\underline{x}_I$ is a large dimension vector, equivalent to long memory [2], the main feature of fractional systems.
- The differential equation is characterized by fractional order n. This parameter is not explicitly used in the model, because it has been converted into four equivalent parameters ω_1', ω_N, α and η. This transformation enables us to simulate the fractional system with a conventional equivalent state-space model and so to estimate its parameters in identification applications [10,12–14,20]. With α and η estimates it is possible to derive n, using $n = 1 - \log\alpha/\log\alpha\eta$.
- Different state-space representations have been derived to simulate fractional systems. A. Oustaloup [19] has proposed an infinite dimensional model based on the numerical approximation of the non-integer derivator. Recently, new models have been introduced; refer for instance to [5,21].

4.3. Comparison with diffusive representation approach

Direct comparison of operator $I_n(s)$ with DR approach (2.2) does not exhibit any particular relation:

we propose to show that these approaches are very close in their principle, but different in their use and potential applications.

Operator $I_n(s)$ can be expanded using ω_i pulsations:

$$I_n(s) = \frac{G_n}{s}\prod_{i=1}^{N}\frac{1+s/\omega_i'}{1+s/\omega_i} = \frac{G_n}{s} + \sum_{i=1}^{N}\frac{c_i}{s+\omega_i}, \quad (42)$$

where

$$c_i = G_n\,\frac{\omega_i-\omega_i'}{\omega_i'}\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{1-\omega_i/\omega_j'}{1-\omega_i/\omega_j}. \quad (43)$$

Inverting relation (42) using Laplace transform, we get the impulse response of the operator:

$$h_n(t) = G_n + \sum_{i=1}^{N} c_i \mathrm{e}^{-\omega_i t} \quad (44)$$

which corresponds to the approximation of fractional integrator (24) using its DR:

$$h(t) = \sum_{i=1}^{N}\mu(\xi_i)\mathrm{e}^{-\xi_i t}\Delta\xi_i, \quad (45)$$

where

$$\mu(\xi_i)\Delta\xi_i = c_i,$$
$$\xi_i = \omega_i. \quad (46)$$

The only difference between $h_n(t)$ and $h(t)$ concerns the supplementary coefficient G_n: is this difference significative?

First, let us interpret the meaning of this coefficient. Consider a fictitious time constant τ and the corresponding mode $G_n\mathrm{e}^{-t/\tau}$. Then

$$G_n = \lim_{\tau\to\infty} G_n\,\mathrm{e}^{-t/\tau} \quad (47)$$

that is to say G_n represents the contribution of an infinite memory mode: G_n avoids a "hard" truncation of ξ_i series when $\xi \to 0$. This was also interpreted initially as the need to link the fractional action to an integer one for $\omega < \omega_1'$.

Secondly, let us show that G_n is essential to the respect of the static gain of the simulated system.

In the previous example, $H(s) = b_0/(a_0 + s^n)$ is approximated by

$$H_n(s) = \frac{b_0 I_n(s)}{1 + a_0 I_n(s)}. \quad (48)$$

Static gain of $H(s)$ is:

$$G_{st} = H(0) = \frac{b_0}{a_0}. \quad (49)$$

Using $H_n(s)$, we get

$$G_{n_{st}} = H_n(0) = \left.\frac{b_0\,\frac{G_n}{s}\prod_{i=1}^{N}\frac{1+(s/\omega_i')}{1+(s/\omega_i)}}{1 + a_0\,\frac{G_n}{s}\prod_{i=1}^{N}\frac{1+(s/\omega_i')}{1+(s/\omega_i)}}\right|_{s=0}$$
$$= \frac{b_0}{a_0}. \quad (50)$$

On the other hand, if we had used the DR of the fractional integrator to perform the simulation in the same way as in our approach, we would have obtained:

$$G_{DR_{st}} = \frac{b_0\sum_{k=1}^{K} c_k/\xi_k}{1 + a_0\sum_{k=1}^{K} c_k/\xi_k} \neq \frac{b_0}{a_0} \quad \forall K. \quad (51)$$

The conclusion of this comparison is that the link between fractional action n and integer one (with the help of G_n/s) is essential to the success of this fractional integrator operator.

On the other hand, because it is impossible to use directly the DR approximation of the fractional operator, it is necessary to compute, prior to the simulation, the DR $\mu(\xi)$ of the desired fractional system, and this is certainly a major constraint for the DR approach.

On the other hand, the only constraint with the operator $I_n(s)$ is the choice of $\{\omega_1', \omega_N\}$ and N appropriate to the simulation of the fractional system. Then, knowledge of n directly refers to α and η parameters, which are fundamentally equivalent to the DR $\mu(\xi)$. Finally, this operator is included in a conventional macroscopic state-vector representation: thus, this approach has large flexibility and it needs no additional knowledge on the fractional system model. Moreover, we will exhibit in the last part that this approach is able to face fractional systems with limited fractional derivation.

5. Numerical simulations

5.1. Benchmark fractional system

5.1.1. Simulations

The modelling of the operator $I_n(s)$ permits to approach the idealfractional integrator when the

frequency domain is very large. For two values of the fractional order n (0.5 and 1.5), step responses of the benchmark fractional system (3) have been simulated using the operator $I_n(s)$ and the direct method described in Section 2.1.

Fig. 6 ($n = 0.5$) and Fig. 7 ($n = 1.5$) show step responses obtained by these two methods with:

- $a_0 = 1$,
- $b_0 = 1$,
- $\omega'_1 = 10^{-5}$ rd/s,
- $\omega_N = 10^5$ rd/s
- a number of cells $N = 30$.

One can notice that in the two cases, responses are superposed. Decreasing the number of cells N and observing the step responses, one can notice that with $N = 10$, small differences between the two responses appear (see Fig. 8). This result is of course evident because the spectral domain is very large (10 decades) and the number of cells used to approximate the fractional integrator is too small; nevertheless, the number of cells necessary to perform a good approximation is quite reasonable (> 10).

The frequency responses of the theoretical fractional system and its approximation using $I_n(s)$ are plotted in Fig. 9 for $n = 0.5$ and Fig. 10 for $n = 1.5$. One can notice that in the frequency domain defined by ω'_1 and ω_N, the two responses coincide. Beyond this domain, the system is equivalent to a first order system in the first case and to a second order system in the second one.

In the case where $N = 10$, the frequency response is plotted in Fig. 11. One can notice the phase oscillations directly linked to the insufficient number of cells.

5.1.2. Conclusion

The direct method gives satisfactory results for any value of n. We obtain the same result with $I_n(s)$ when the number of cells is sufficiently large. We have not presented simulation results for the Diffusive Representation approach but as expected, they are also satisfactory but only for $0 < n < 1$.

We have already exhibited the limitations of the DR approach. On the contrary, the direct one applies to any kind of fractional systems: the essential drawback is the increasing dimension of the sums directly related to the numerical approximation (1). This is a real

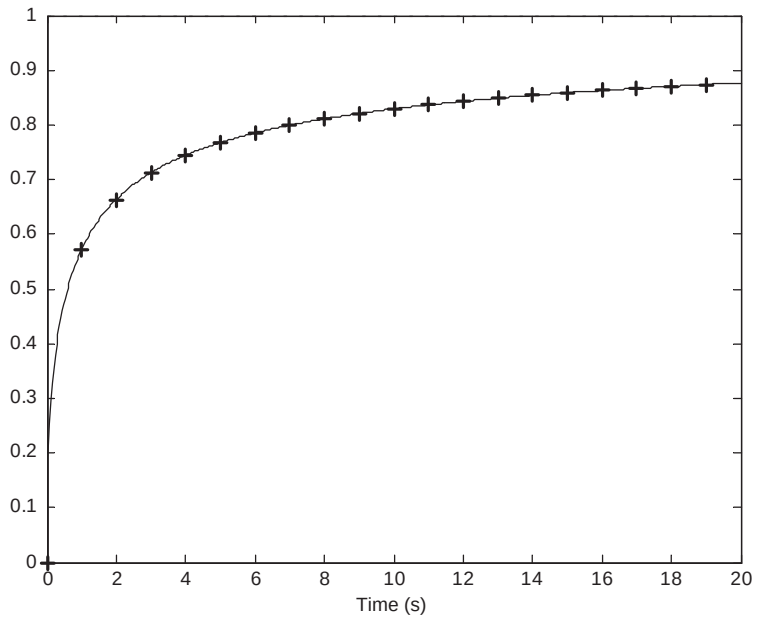

Fig. 6. Step response for $n = 0.5$ (—, simulation with $I_n(s)$; +, simulation with the direct method).

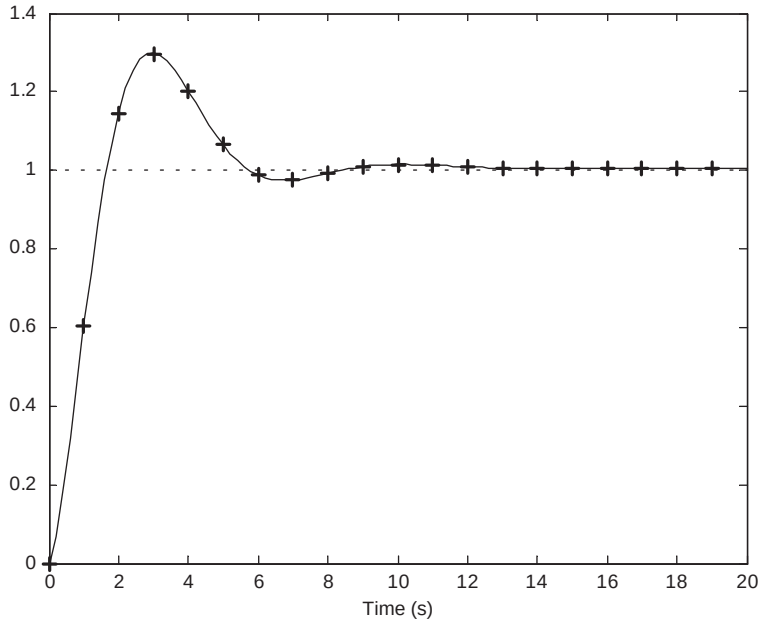

Fig. 7. Step response for $n = 1.5$ (—, simulation with $I_n(s)$; +, simulation with the direct method).

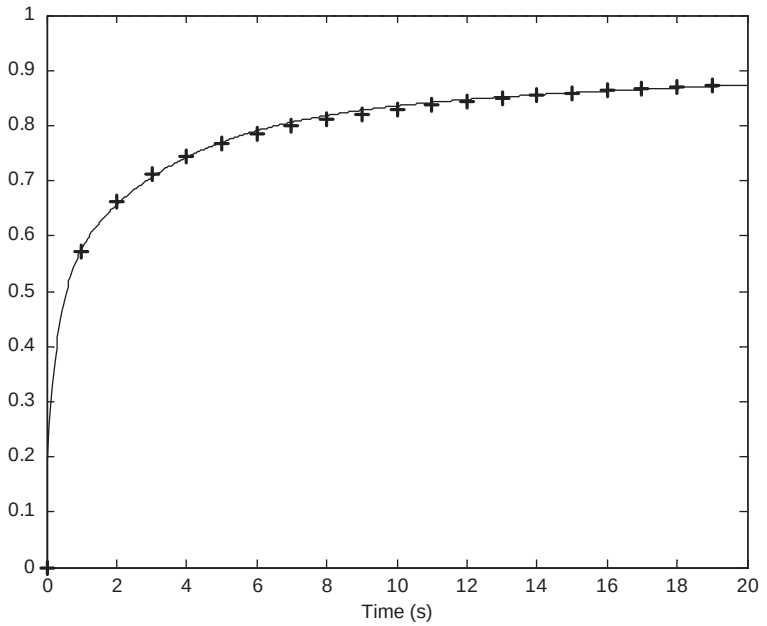

Fig. 8. Step response for $n = 0.5$ and $N = 10$ (—, simulation with $I_n(s)$; +, simulation with the direct method).

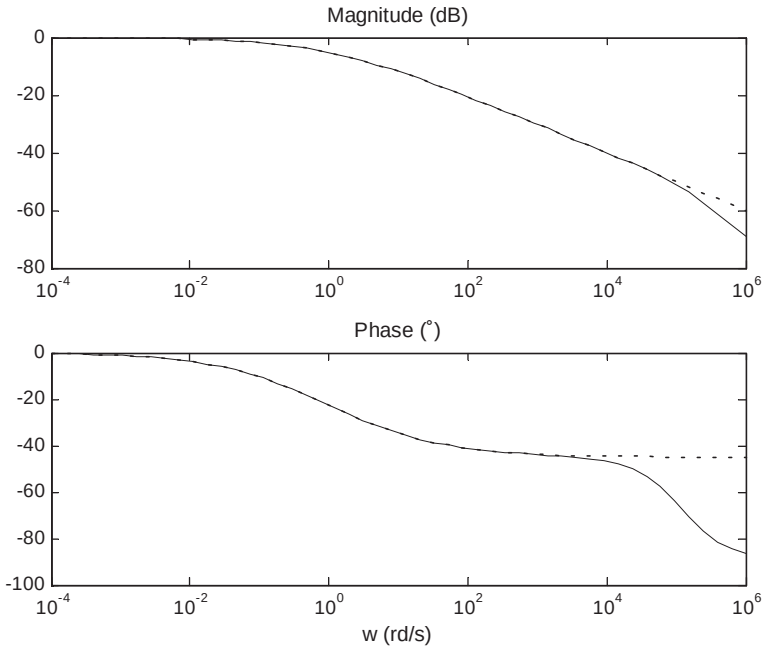

Fig. 9. Bode plot for $n = 0.5$ and $N = 30$ (—, approximated system; ..., theoretical system).

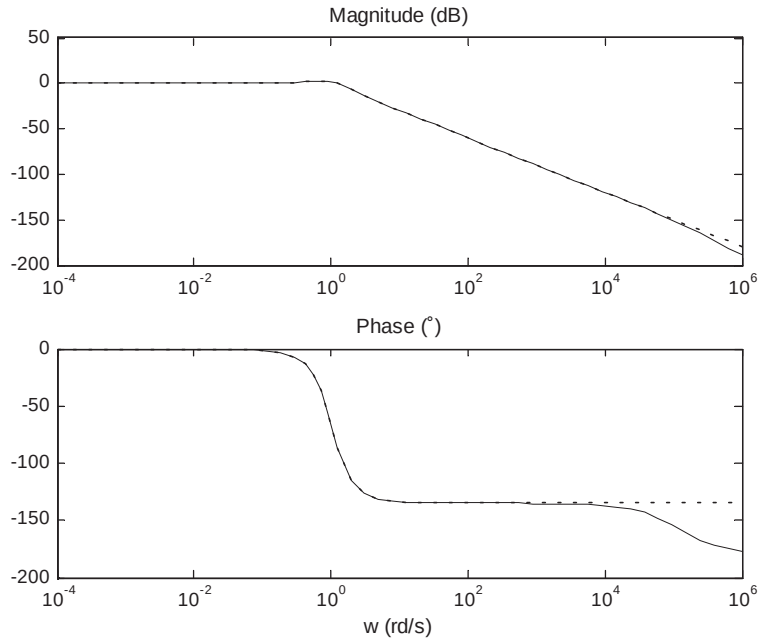

Fig. 10. Bode plot for $n = 1.5$ and $N = 30$ (—, approximated system; ..., theoretical system).

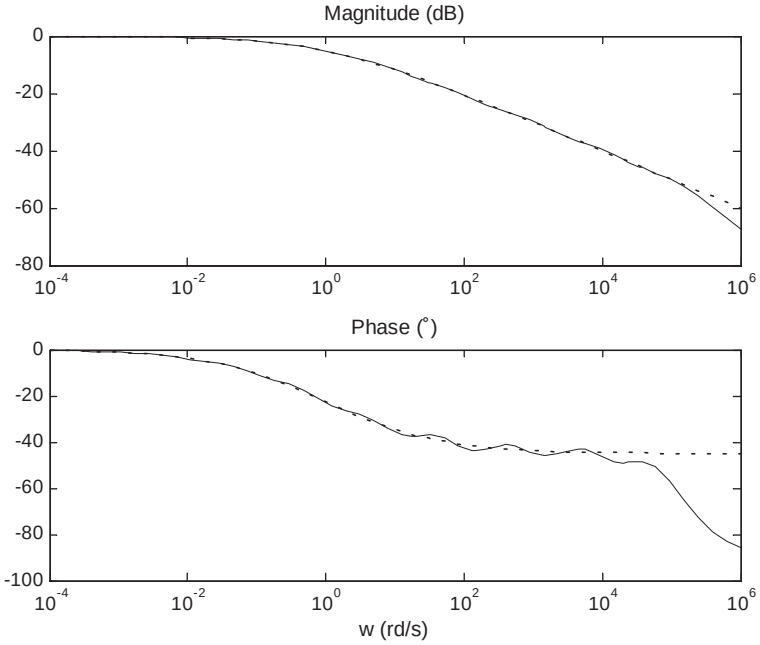

Fig. 11. Bode plot for $n = 0.5$ and $N = 10$ (—, approximated system; ..., theoretical system).

constraint when this approximation is used to simulate the model in identification and parameter estimation [4].

5.2. Simulation of a fractional system with bounded spectral range

When the frequency domain where acts the non-integer order is not infinite but limited to some decades, the operator defined in this article permits nevertheless to give corresponding time and frequency responses.

Consider the system $H(s)$:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0 + b_1 s^n}{a_0 + a_1 s^n + s^{n+1}}, \tag{52}$$

where the fractional order n is limited to one decade.

Its state-space model is represented in Fig. 12:

For this example, the following parameters have been chosen:

- $a_0 = 1$, $a_1 = 1$, $b_0 = 1$, $b_1 = 1$,
- $n = 0.4$.

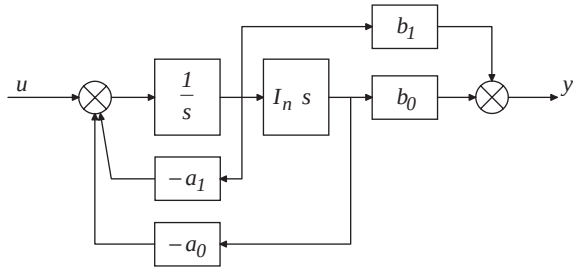

Fig. 12. State-space representation of the system (52).

Two simulations are performed. The first one (solid line) is done with a fractional integrator limited to one decade with $N=10$ cells, $\omega_b=0.1$ rd/s and $\omega_h=1$ rd/s. The second simulation (dotted line) is performed with an ideal fractional operator with the parameters $N=30$ cells, $\omega_b = 10^{-5}$ rd/s and $\omega_h = 10^5$ rd/s.

Figs. 13 and 14 exhibit respectively the step responses of $H(s)$ and its bode plots in the two cases.

This last example exhibits clearly that the proposed fractional integrator operator is able to face conventional fractional modelling and simulation as well as non-conventional ones, like fractional derivation with

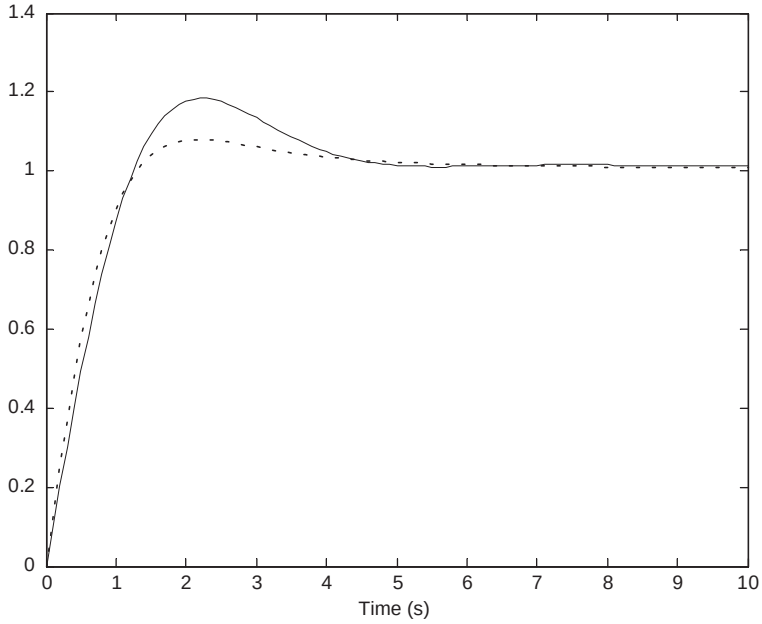

Fig. 13. Step responses of the system.

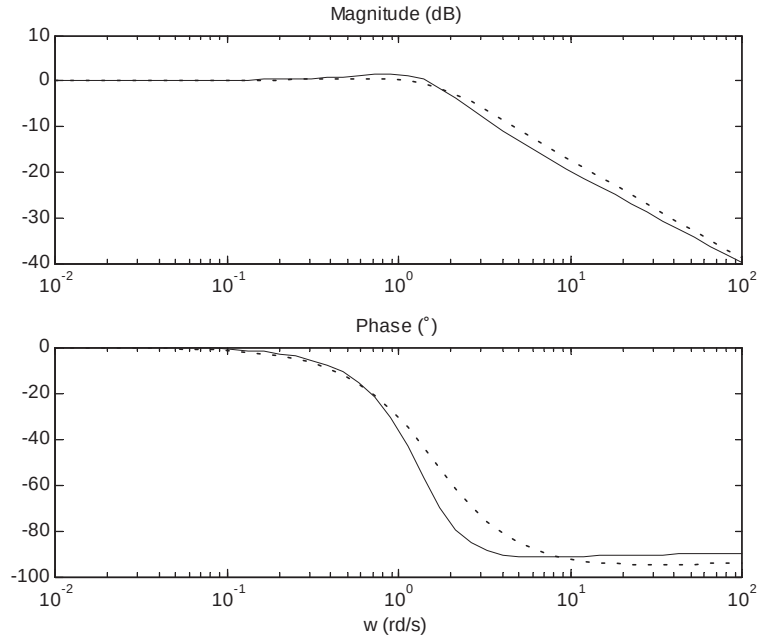

Fig. 14. Bode plots of the system.

limited spectral range. It is also evident that neither direct numerical approximation nor diffusive representation can take into account this non-conventional problem.

6. Conclusion

An original method for modelling and simulation of fractional systems has been presented in this article. This modelling is based on a new fractional integrator operator, associated to a N dimensional state-space representation. A few parameters used for the design of the non-integer action and its spectral range are necessary to characterize this operator.

Theoretical and numerical comparisons with other techniques commonly used for the simulation of fractional systems have exhibited the performances of this original approach. Its main interest is to propose a general framework for the modelling of fractional systems based on a macro state-space representation, where conventional integration is replaced by fractional one with the help of the integrator operator. Another important feature of this new approach is its flexibility because it applies to different types of fractional models and to non-conventional non-integer derivation with limited spectral range.

This fractional operator has been already applied to identification problems with output error technique. Various experiments [10,12–14,20] have confirmed the interest and the validity of this new approach for modelling and simulation of fractional systems in diffusive applications.

References

[1] J.L. Battaglia, Méthodes d'identification de modèles à dérivées d'ordres non entiers et de réduction modale. Application à la résolution de problèmes thermiques inverses dans des systèmes industriels, Habilitation à Diriger des Recherches, Université de Bordeaux I, 2002.

[2] J. Beran, Statistical methods for data with long-range dependance, Stat. Sci. 7 (1992) 404–427.

[3] Y.Q. Chen, K.L. Moore, Discretization schemes for fractional differentiators and integrators, IEEE Trans. Circuits System 1: Fundamental Theory Appl. 49 (3) (2002) 363–367.

[4] O. Cois, Systèmes linéaires non entiers et identification par modèle non entier: application en thermique, Thèse de Doctorat, Université de Bordeaux I, 2002.

[5] M. Fliess, R. Hotzel, Linear systems with a derivation of non integer order: introductory theory with an example, Proceedings CESA'96 Symposium on Models, Analysis and Simulation, Vol. 1, France, 1996, pp. 213–217.

[6] D. Heleschewitz, D. Matignon, Diffusive realisations of fractional integrodifferential operators: structural analysis under approximation, Proceedings IFAC Conference System, Structure and Control, Vol. 2, Nantes, France, 1998, pp. 243–248.

[7] M. Ichise, Y. Nagayanagi, T. Kojima, An analog simulation of non integer order transfer functions for analysis of electrode processes, J. Electroanal. Chem. Interfacial Electrochem. 33 (1971) 253.

[8] L. Le Lay, A. Oustaloup, J.-C. Trigeassou, F. Levron, Frequency identification by non integer model, Proceedings IFAC Conference System, Structure and Control, Nantes, France, 1998, pp. 297–302.

[9] L. Le Lay, A. Oustaloup, J.-C. Trigeassou, F. Levron, Frequency identification by implicit differentiation models, Proceedings AVCS'98 International Conference on Advance in Vehicle Control and Safety, Amiens, France, 1998, 351–356.

[10] J. Lin, Modélisation et identification de systèmes d'ordre non entier, Thèse de Doctorat, Université de Poitiers, 2001.

[11] J. Lin, T. Poinot, Modélisation de systèmes d'ordre non entier, JDA'99 Journées Doctorales d'Automatique, Nancy, France, 1999, pp. 53–56.

[12] J. Lin, T. Poinot, J.C. Trigeassou, P. Coirault, Parameter estimation of fractional systems. Application to heat transfer, ECC 2001, European Control Conference, Porto, Portugal, 2001, pp. 2644–2649.

[13] J. Lin, T. Poinot, J.C. Trigeassou, H. Kabbaj, J. Faucher, Modélisation et identification d'ordre non entier d'une machine asynchrone, CIFA'2000, Conférence Internationale Francophone d'Automatique, Lille, France, 2000.

[14] J. Lin, T. Poinot, J.C. Trigeassou, R. Ouvrard, Parameter estimation of fractional systems: application to the modeling of a lead-acid battery, SYSID 2000, 12th IFAC Symposium on System Identification, USA, 2000.

[15] D. Matignon, B. d'Andréa-Novel, P. Depalle, A. Oustaloup, Viscothermal losses in wind instrument: a non integer model, in: System and Networks: Mathematical Theory and Applications, Vol. 2, Akademie Verlag, Berlin, 1994.

[16] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley-Interscience Publication, New York, 1993.

[17] G. Montseny, Diffusive representation of pseudo-differential time-operators, Proceedings Fractional Differential Systems: Models, Methods and Applications, Paris, 1998.

[18] A. Oustaloup, Systèmes asservis linéaires d'ordre fractionnaire, Masson, Paris, 1983.

[19] A. Oustaloup, La dérivation non entière: théorie, synthèse et applications, Hermès, 1995.

[20] T. Poinot, J.C. Trigeassou, Parameter estimation of fractional models: application to the modeling of diffusive systems, 15th IFAC World Congress, Barcelona, Spain, 2002.
[21] H.F. Raynaud, A. Zergainoh, State-space representation of fractional linear filters, Proceedings IFAC Conference System Structure and Control, Nantes, France, 1998.
[22] N. Retière, M. Ivanes, Modeling of electric machines by implicit derivative half-order systems, IEEE Power Eng. Rev. 18 (9) (1998) 62–64.
[23] J.A. TenReiro Machado, Discrete-time fractional-order controllers, Proceedings IEEE-INES'2000, Fourth International Conference on Intelligent Engineering Systems, Slovenia, 2000.